INTRODUCTION

A. DERIVATION OF THE CONCEPT OF PURE MATHEMATICS

1. The principal division of the sciences is into the real and the formal. The real represent the existent in thought as existing independently of thought, and their truth consists in the correspondence of the thought with that existent. The formal on the other hand have as their object what has been produced by thought alone, and their truth consists in the correspondence between the thought processes themselves.

Thought exists only in reference to an existent that confronts it and is portrayed by the thought; but in the real sciences this existent is independent, existing for itself outside of thought, whereas in the formal it is established by thought itself, when a second thought process is confronted as an existent. Now if truth is in general based on the correspondence of the thought with the existent, then in particular in the formal sciences it is based on the correspondence of the second thought process with that existent established by the first, that is, on the correspondence of the two thought processes. Thus proof in the formal sciences does not extend beyond the sphere of thought, but resides purely in the combination of different thought processes. Consequently, the formal sciences cannot begin with postulates, as do the real; rather, definitions comprise their foundation.*

2. The formal sciences treat either the *general* laws of thought or the

*Although postulates have been introduced into the formal sciences, for example in arithmetic, this is to be regarded as an error, only to be explained by the corresponding treatment of geometry. I will return later to consider this in more detail. Here it is enough to have demonstrated that postulates are necessarily absent from the formal sciences.

particular as established by means of thought, the former being the dialectic (logic),* the latter, pure mathematics.

The contrast between the general and the particular thus produces the division of the formal sciences into dialectics and mathematics. The first is a philosophical science, since it seeks the unity in all thought, while mathematics has the opposite orientation in that it regards each individual thought as a particular.

3. Pure mathematics is therefore the science of the *particular* existent that *has come to be* by thought. The particular existent, viewed in this sense, we call a thought form or simply a *form;* thus pure mathematics is the *theory of forms.*

The name "theory of magnitude" is inappropriate for all of mathematics, since one finds no use for magnitude in a substantial branch of it, namely combination theory, and even in arithmetic only in an incidental sense.** On the other hand the expression "form" might seem rather too broad, and the name "thought form" more appropriate; but the form in its pure meaning, devoid of all real content, is precisely nothing but the thought form, and thus the expression is suitable.

Before we proceed to the division of the theory of forms we have to separate out one branch that has hitherto incorrectly been included in it. This branch is geometry. From the concepts set out above it is evident that geometry, like mechanics, refers to a real existent; for geometry, this is space. This is clear since the concept of space can in no way be produced by thought, but rather emerges as something given. Anyone who would maintain the contrary must undertake the task of deducing the necessity for the three dimensions of space from the laws of pure thought, a problem whose solution is patently impossible.

One who, despite being obliged to admit this, prefers to extend the name "mathematics" to geometry may indeed do so if in return he allows us our name "theory of forms" or its equivalent; but we must point out to him that his name, in being made to comprehend too much, must ultimately be abandoned as superfluous.

The position of geometry relative to the theory of forms depends on the relation of space perception to pure thought. Although just now we

*Logic has a mathematical aspect that one can call formal logic, whose content has been developed jointly by my brother Robert and myself, and is presented in an original form by him in his second book, the *Formenlehre,* Stettin: 1872. (1877) New edition in 2 vols.: *Logik u. Formenlehre,* Stettin: 1890, 1891.

**The concept of magnitude is replaced by that of number in arithmetic; language thus differentiates very well "to add" and "to subtract" as pertaining to number, "to increase" and "to decrease" to magnitude.

said that that perception confronts thought as something independently given, it is not thereby asserted that space perception emerges only from the consideration of solid objects; rather, it is that fundamental perception imparted to us by the openness of our senses to the sensible world, which adheres to us as closely as body to soul. It is the same with time and with the perception of motion based on time and space, wherefore one could count the pure theory of motion (phorometry) among the mathematical sciences with as much justice as geometry. The concept of motive force flows from the idea of motion through the contrast between cause and effect. Thus geometry, phorometry, and mechanics appear as applications of the theory of forms to the fundamental perceptions of the sensible world.

B. DERIVATION OF THE CONCEPT OF EXTENSION THEORY[4]

4. Each particular existent brought to be by thought (cf. no. 3) can come about in one of two ways, either through a simple act of *generation* or through a twofold act of *placement and conjunction.* That arising in the first way is the *continuous form,* or *magnitude* in the narrow sense, while that arising in the second way is the *discrete* or *conjunctive form.*

The simple act of becoming yields the continuous form. For the discrete form, that posited for conjunction is of course also produced by thought, but for the act of conjunction it appears as given; and the structure produced from the givens as the discrete form is a mere correlative thought. The concept of continuous becoming is more easily grasped if one first treats it by analogy with the more familiar discrete mode of emergence. Thus since in continuous generation what has already become is always retained in that correlative thought together with the newly emerging at the moment of its emergence, so by analogy one discerns in the *concept* of the continuous form a twofold act of placement and conjunction, but in this case the two are united in a single act, and thus proceed together as an indivisible unit. Thus, of the two parts of the conjunction (temporarily retaining this expression for the sake of the analogy), the one has already become, but the other newly emerges at the moment of conjunction itself, and thus is not already complete prior to conjunction. Both acts, placement and conjunction, are thus merged together so that conjunction cannot precede placement, nor is placement possible before conjunction. Or again, speaking in the sense appropriate for the continuous, that which newly emerges does so precisely upon that which has already become, and thus, in that moment of becoming itself, appears in its further course as growing there.

The opposition between the discrete and the continuous is (as with all true oppositions) fluid, since the discrete can also be regarded as continuous, and the continuous as discrete. The discrete may be regarded as continuous if that conjoined is itself again regarded as given, and the act of conjunction as a moment of becoming. And the continuous can be regarded as discrete if every moment of becoming is regarded as a mere conjunctive act, and that so conjoined as a given for the conjunction.

5. Each particular existent becomes such through the concept of the *different,* whereby it is coordinated with other particular existents, and through this with the *equal,* whereby it is subordinated to the same universals with other existents. That arising from the equal we may call the *algebraic form,* that from the different the *combinatorial form.*

The opposition between equal and different is also fluid. Equals are already different insofar as the one and the other equal to it are at all separate (but without this separation there would be only a unity, and thus no equality); two different ideas are already equal insofar as they are conjoined by the activity of relating them, that is both are equally conjuncts. However the two concepts by no means thereby blend together so that one could define a scale with which to specify how much equality obtains between two ideas, and how much difference; rather, although the different always somehow adheres to the equal and conversely, at the moment of consideration only the one is encompassed, while the other only appears as the requisite basis of the first.

By the algebraic form is here understood not only number but also that corresponding to number in the continuous domain, and by the combinatorial form not only combination but also its correspondent in the continuous.

6. From the interaction of these two oppositions, the former of which is related to the type of generation, the latter to the elements of generation, arise the four species of form and the corresponding branches of the theory of forms; thus the discrete form thereby separates into number and combination. *Number* is the algebraic discrete form; that is, it is the unification of those established as equal. *Combination* is the combinatorial discrete form; that is, it is the unification of those established as different. The sciences of the discrete are therefore *number theory* and *combination theory* (relation theory).

It scarcely needs further demonstration that the concept of number is hereby completely exhausted and precisely delimited, and likewise that of combination. And since the oppositions from which these definitions follow are the most elementary imparted directly in the concept of the mathematical form, the above derivation is indeed thereby sufficiently justified.* I note in addition only that this opposition between the two forms is expressed very clearly by the different notations for their elements, since that conjoined to number is designated by a single symbol (1), that conjoined to combination by different ones from among the remaining completely arbitrary symbols (the letters of the alphabet). —That each set of things (particular existents) can be interpreted as number as well as combination scarcely requires mention.

7. In precisely the same way, the continuous form or magnitude separates into the algebraic continuous form or *intensive magnitude* and the combinatorial continuous form or *extensive magnitude.* The intensive magnitude is thus that arising through generation of equals, the extensive magnitude or *extension* that arising through generation of the different. As variable magnitudes the former constitute the foundation of function theory, that is differential and integral calculus, the latter the foundation of extension theory.

The first of these two branches is usually regarded as subordinate to number theory, a higher branch; the second however appears to be a previously unknown branch, whence it is necessary to explain this difficult conception through the notion of continuous flow.

As with number there prevails the unification of that imagined together, and with combination the separation; so also with the intensive magnitude there appears the unification of elements, indeed separate conceptually, but which form the intensive magnitude only in their essential equality. In contrast, with the extensive magnitude there prevails the separation of elements that are indeed unified insofar as they form a single magnitude, but which constitute that magnitude precisely in their mutual separation. It is thus somewhat as if the intensive magnitude is number become fluid, the extensive magnitude combination become fluid. The latter is essentially a proceeding of elements mutually apart, retaining them as being mutually parted. With it, the generating element appears as changing, that is as passing through a variety of states, the collection of these various states forming precisely the domain of the extensive magnitude. With the intensive magnitude, its generation produces a series of states equivalent to itself, whose quantity is precisely the intensive magnitude. The best example we can offer for the extensive magnitude is the line segment (displacement), whose elements proceed essentially apart from each other and thus constitute precisely the line as extension; on the other hand, an

*The concepts of number and combination were developed in a completely similar way seventeen years ago in a paper written by my father on the concept of pure number theory, published in the *Programme* of the Stettin Gymnasium for 1827, without however coming to the attention of a wider public.

example of the intensive magnitude is perhaps a point associated with a specific force, since in this case the elements are not removed, but rather are presented only as an intensity, thus forming a specific order of intensification.

In this case the difference so established is also expressed beautifully in the notation; thus with intensive magnitudes, which comprise the subject matter of function theory, one does not distinguish the elements by different symbols, but rather, where a particular symbol appears the complete variable magnitude is represented. On the other hand, with extensive magnitudes, or with their concrete realizations (lines) different elements are designated by different symbols (letters), just as in combination theory. It is also clear that each real magnitude can be viewed in two ways, both as intensive and extensive: thus lines may also be regarded as intensive magnitudes if one removes from their nature the way their elements lie apart, and retains simply the quantity of their elements; and in the same way a point associated with a specific force can be thought of as an extensive magnitude, since one can represent the force in the form of a line.

Historically, among the four branches of mathematics the discrete appeared earlier than the continuous (since the former is closer to the analytical sense than the latter), the algebraic earlier than the combinatorial (since the equal is easier to grasp than the different). Thus number theory is the earliest, combination theory and differential calculus appear simultaneously, and extension theory in its abstract form is necessarily last of all, although its concrete (but limited) image, the theory of space, already belonged to the earliest time.

8. Antecedent to the division of the theory of forms into its four branches is a more general subject that we may call the general theory of forms. In it are presented the general conjunctive laws that apply to all branches alike.

This preliminary subject is not intended simply to save repeating the same material in all four branches and thus to condense the treatment of the different parts, but also permits what naturally belongs together to appear together, and acts as the foundation of the whole.

C. EXPOSITION OF THE CONCEPT OF EXTENSION THEORY

9. Continuous becoming, analyzed into its parts, appears as a continuous production with retention of that which has already become. With the extensive form, that which is newly produced is always defined as different; if, during this process, we no longer always retain what has already become, then we arrive at the concept of *continuous evolution.* We call that which undergoes this evolution the generating element, and the

generating element, in any of the states it assumes in its evolution, an element of the continuous form. Accordingly, the extensive form is the collection of all elements into which the generating element is transformed by continuous evolution.

The concept of continuous evolution can only arise with extensive magnitudes; with the intensive magnitudes, always dropping that which has already become would leave only the completely empty continuous tendency to become.

In the theory of space, the point appears as the element, the evolving locus or motion as its continuous evolution, and the various positions of the point in space as its different states.

10. The different must be produced according to a law if it is to have a specific resultant. For the elementary form this law must be the same for all moments of becoming. The *elementary* extensive form is thus the form that results from an evolution of the generating element according to this same law; the collection of all elements generated according to the same law we call a *system* or a *domain.*

Since that different from a given may be any one of an infinite manifold, it would become completely indeterminate were it not constrained by a fixed law. This law is, however, not at all defined by the content of the pure theory of forms; rather, the concept of extension is defined by the purely abstract idea of lawfulness, and the concept of the *elementary* extension by the same law for every moment of evolution. Accordingly the elementary extension has the property that if, from an element a of the extension there results another element b of the same extension, then by an identical process a third element c of it results from b.

In space the theory of the uniformity of direction is that single law governing the individual evolutions; thus the displacement represents the elementary extension in space theory, and the infinite straight line the whole system.

11. If one uses two different laws of evolution, the collection of elements generated by the two laws is a system of second order. The laws of evolution by which the elements of this system are produced from each other depend only on these two laws; if one adds a third independent law one arrives at a system of third order, and so on.

The theory of space may again serve as an example. Here the collection of elements of a plane are generated from a single element together with two directions when the generating element progresses by arbitrary amounts in the two directions, and the totality of points (elements) so generated are collected together as a single object. The plane is thus the system of second order; in it there is an infinite set of directions dependent on those two original directions. If a third independent direction is added, then by means of this direction, the whole of infinite space (as the system of third order) is produced. In this example one cannot proceed beyond three independent directions (evolutionary laws); but in pure extension theory their number can be infinitely increased.

12. The more precise determination of the difference between the laws requires a method of generation by means of which one system transforms into another. This transformation of different systems into one another thus forms a second natural order in the domain of extension theory, and the elementary presentation of this science thus concludes with it.

Rotational motion corresponds to this transformation of one system into another in the theory of space; and connected with this are angular magnitude, absolute length, orthogonality, and so forth. All of this will be worked out, but only in the second volume of the *Ausdehnungslehre.*

D. FORM OF PRESENTATION

13. The philosophical method characteristically proceeds by oppositions, and thus progresses from the general to the particular, whereas the mathematical method proceeds from the most elementary concepts to the more complex, and thus produces new and more general concepts by the conjunction of particulars.

Thus whereas in the former the overview of the whole predominates, and its development consists precisely in the gradual ramification and articulation of the whole, in the latter the interconnection of particulars is emphasized, and separate, independent developments combine together, each becoming only a factor in the following concatention. This difference in method is implicit in their concepts; for in philosophy the primitive is precisely the unity of the idea, the particular being derived, whereas in mathematics the particular is the primitive, the unifying idea, the last aspired to; and thus are caused their opposing developments.

14. Since mathematics as well as philosophy are sciences in the strict sense, the methods of both must have something in common that makes them scientific. Now we characterize a method of treatment as scientific if the reader is thereby on the one hand led necessarily to recognize the individual truths, and on the other is placed in a position from which he can survey each point in the broader sweep of the development.

Everyone admits the indispensability of the first requirement, that is, of scientific rigor. What the second entails, most mathematicians do not yet admit as a valid point. Thus there often is found a proof in which, were the conclusion not stated at the beginning, one would understand nothing of where it is leading or how, and only after an extended period of imitating each step blindly and as it were randomly would one suddenly, and before one realized it, arrive at the truth sought. Such a proof may lack nothing in point of rigor, but it is not scientific; for it lacks the second requirement, that of the clarity of an overview. Whoever therefore imitates such a proof does not arrive at a clear cogniton of the truth but, if he does not subsequently produce that overview himself, remains completely dependent on that particular method by which he found that truth established. And the feeling of confinement, which in such cases arises at least during the acquisition of the method, is most oppressive for one accustomed to thinking freely and automatically, and to acquiring all he absorbs spontaneously and vividly. On the other hand, if the reader is placed in a position such that at each point of the development he can see where it is going, then he remains master of the material, is no longer bound to the particular form of presentation, and that which is assimilated is a faithful reproduction.

15. For each given part of the presentation, the nature of its further development is essentially fixed by a dominant idea that is either nothing more than a supposed analogy with cognate and already familiar branches of knowledge, or, and this is the better case, is a direct intuition of the next succeeding truth.

As it plays into a cognate domain, the analogy is only a makeshift if it is not really important to emphasize throughout the relation to a cognate branch and thus continually to draw the analogy with it.* Intuition might seem to be alien to the domain of pure science, and most of all to mathematics; but without it, it is impossible to discover any new truth. One cannot be led to a desired result by blind combinations; rather, what to combine and how must be determined by the dominant idea, and again this idea can, before it has materialized in the science itself, only appear in the form of an intuition. Intuition is therefore nearly indispensable in the domain of science. In particular, if it is of the right sort, it is what in an overall view of the entire course of development leads to the new truth, but not yet at an opportune moment for its development, and thus initially only as a dim presentiment; the detailing of that moment includes both the discovery of the truth and the critique of the presentiment.

*In the science treated here this occurs in its relation to geometry, on which I have mainly drawn by way of analogy.

16. The essence of a scientific presentation is thus an interlocked pair of approaches, one of which leads consistently from one truth to another and forms the actual content, while the other controls the treatment and thus determines the form. In mathematics these two separate lines of development diverge most acutely.

Following the model originated by Euclid, it has long been the practice in mathematics to present only the approach that comprises the actual content; the reader is left to puzzle out the other by reading between the lines. However perfect both its order and presentation may be, for someone wishing to learn the science it is impossible thus to maintain an overview at each point, and to be placed in a position to proceed freely and automatically. It is thus the more necessary that, as far as possible, the reader be placed in the position in which a discoverer of the truths is certain to be found in the best cases. In these circumstances, there occurs a continual reflection about the course of development as the truths are discovered; a characteristic line of thought develops in the reader about the procedure followed and about the idea lying at the foundation of the whole; and this line of thought forms the true nucleus and spirit of his activity, while the consistent detailing of the truths is only the embodiment of that idea.

As a consequence of the method of presentation, the reader may then expect to be able to progress independently, without having to be guided along such lines of thought, regarding himself as independent of the discoverer of the truth and thus reversing the relation between himself and the author, whereby the whole composition of the work appears superfluous. Accordingly, the modern mathematicians, and particularly the French, have begun to weave the two approaches together. The attraction of their works consists precisely in that the reader feels free and is not constrained to forms he must slavishly follow while not being fully conversant with them.

That these approaches diverge most acutely in mathematics originates in the characteristics of its method (no. 13); that is, since it progresses by the connection of particulars, the unity of the idea coming at the end. The second approach has a character completely opposite to the first, and their interweaving appears more difficult in mathematics than in any other science. One ought not, however, abandon the attempt on account of this difficulty, as often happens with German mathematicians.

In the present work I have therefore adopted the method suggested, and this seems to me particularly necessary with a new science, as the idea is itself simultaneously being brought to light.

SURVEY OF THE GENERAL THEORY OF FORMS

§ 1. Concept of Equality

By the general theory of forms we mean that series of truths that relate to all branches of mathematics in the same way, and which thus assume only the general concepts of equality and difference, conjunction and separation. The general theory of forms must therefore precede all special branches of mathematics.* Since however that general branch does not yet exist as such, and we cannot omit it without entangling ourselves in lengthy digressions, we have no choice but to develop this subject to the extent required for our science.

Here we first establish the concepts of equality and difference.

Since two equals must appear as different in order to stand out as two, and two differents must appear as different aspects of equals,** it seems necessary from a superficial consideration to formulate various relations of equality and difference. Thus for example in comparing two line segments one can assert equality of direction or length, of direction and length, of direction and position, and so forth; and in comparing other things, further relations of equality emerge. But already the fact that these relations change according to the character of the things being compared is proof enough that these relations do not belong to the concept of equality itself, but to the objects to which this concept of equality is applied. Thus, for example, for two equally long displacements we cannot say that they are equal as such, but only that their lengths are equal, and so these lengths themselves stand in the absolute relation of equality. Thus we have preserved the simplicity of the concept of equality and can define it: *Those are equal of which one can always assert the same, or more generally what in any judgment can be substituted one for the other.****

It plainly follows from this that, if two forms are each equal to a third, they are also equal to each other; and that those generated in the same way out of equals are again equal.

*Cf. intro., no. 8.
**Cf. intro., no. 5.
***This is not intended as a philosophical definition, but only as an understanding of the word, so that it will not be taken in different ways. The philosophical definition would have to take up the opposition of equal and different in its fluidity and its strict limits, for which a considerable apparatus of definitions not belonging here would be required.

§ 2. Concept of Conjunction

The second opposition whose consideration we have to take into account here is that of conjunction and separation. If two magnitudes or forms (which name we prefer, cf. intro., no. 3) are conjoined, they are called the *factors* of the conjunction; the form that is produced by the conjunction of the two, the *product* of the conjunction. If the two factors need distinguishing, we call the one the *prefactor,* the other the *postfactor.*

As the general sign for conjunction we take the symbol $\cap$; now if a and b are the factors, with a the prefactor, b the postfactor, then we indicate the product of their conjunction as $(a \cap b)$, where the parentheses here express that the conjunction indicates that the factors are no longer separate, but that their concepts are unified.* The product of the conjunction can be further conjoined with another form, whence one proceeds to a conjunction of several factors, which however initially appears as just the sequential conjunction of two at a time. For our convenience we use the usual abbreviated parenthetical notation, whereby we omit the parentheses around two conjoined symbols if the opening parenthesis [(] stands at the beginning of the whole expression, or follows after another open parenthesis; for example, instead of $((a \cap b) \cap c)$ we write $a \cap b \cap c$.

§ 3. Combinability of Factors[5]

The particular type of conjunction will now be specified by what maintains the same product, that is under what circumstances and to what extent the product remains equal to itself.

The only changes one can undertake without changing the individual forms conjoined are a change in the parentheses and a reordering of factors. We consider first the conjunction for which, in a product of three factors, the placement of the parentheses makes no real difference; that is to say, no difference in the product is found. In this case we have $a \cap (b \cap c) = a \cap b \cap c$, from which it follows at once that the parentheses may also be omitted in every multifactor conjunction of this type without changing its product. Thus first of all, by virtue of the above assumption, each pair of parentheses enclose a two-factor expression, and this expression must itself be combined as a factor with another form; briefly, it appears as a combination of three forms for which we assume that one can omit the parentheses without changing the product of their conjunction. Thus, since one can replace each form with its equal, it follows that the

*How this unification is effected, and what is thereby imposed in each case on the idea of the individual conjunctions, depends on the nature of the particular type of conjunction.

combined product is not changed by omitting the parentheses. Therefore:

If a conjunction is of the type that the parentheses may be omitted for three factors, then the same is true for any number of factors.

Since, according to the above theorem, one is permitted to omit the parentheses in each of two expressions that differ only in the placement of their parentheses, the two are equal since they are both equal to the same expression (without parentheses); one thus has the foregoing theorem in a somewhat more general form:

If a conjunction is of the type that the placement of parentheses is of no real consequence for three factors, then the same is also true for any number of factors.

§ 4. Interchangeability of Factors. Concept of Elementary Conjunction

Now if, on the other hand, only the interchangeability of the two factors of a conjunction were established, then no further conclusion could be drawn. On adding this assumption to that made in the previous paragraph, however, it follows that the ordering of factors is also unimportant for the combined product in multifactor expressions, since one can then easily show that the interchange of any two factors in the expression is permitted.

Thus, according to the theorem previously established (§ 3), one can enclose two factors, whose interchangeability is known, in parentheses without changing the combined product, interchange the two factors without changing the product of their conjunction (by assumption), and thus without changing the product of the whole conjunction (since one can replace each form with its equal), and finally the parentheses can be restored to their original positions. Thus we establish the interchangeability of two adjacent forms. Since however one can now continue this process to bring each factor to any arbitrary position, the order of factors is necessarily unimportant. Combining this result with that of the preceding paragraph:

If a conjunction is of the type that one can arbitrarily place parentheses around any three of its factors and change the order of any two of its factors without changing the product, then it is also true that the placement of parentheses and the order of factors is unimportant for the product of any number of factors.

For brevity we call a conjunction satisfying the given conditions *elementary.* Without referring to the nature of the forms conjoined, a further specification of the nature of the conjunction is not possible. We therefore turn to the inversion of the form obtained, that is to the analytic process.

§ 5. The Synthetic and Analytic Conjunctions

The analytic process consists in seeking one factor of a conjunction in terms of its product and the other factor. Thus to each conjunction there belong two types of analytic process, one for the prefactor and one for the postfactor. They both yield the same product if the two factors of the original conjunction are interchangeable. Since this analytic process can also be regarded as a conjunction, we distinguish the original or *synthetic* conjunction from the inverse or *analytic* conjunction.

In the following we suppose the synthetic conjunction is elementary in the sense of the preceding paragraph, and retain for it the symbol $\cap$; in this case the two general inverses merge into one, and we adopt the inverted symbol $\cup$ for the corresponding analytic conjunction, so that we make the prefactor of the synthetic conjunction that which is given by the analytic.

Accordingly $a \cup b$ denotes that form which, conjoined synthetically with b, gives a, that is $a \cup b \cap b = a$. From this there follows at once that $a \cup b \cup c$ denotes that form which gives a when it is synthetically conjoined with c and then with b, and therefore according to § 4 also that form which gives a when conjoined synthetically with the same values in inverted order, or with $b \cap c$, thus:

$$\begin{aligned} a \cup b \cup c &= a \cup c \cup b \\ &= a \cup (b \cap c); \end{aligned}$$

and since the same result is true for any number of factors, it follows that the order of factors that are preceded by analytic symbols is also not important, and that one may enclose these factors in parentheses, provided one inverts the enclosed symbols. Furthermore it follows that

$$a \cup (b \cup c) = a \cup b \cap c.$$

Thus from the definition of the analytic conjunction one has

$$a \cup (b \cup c) = a \cup (b \cup c) \cup c \cap c;$$

and this expression is again, by virtue of the laws just established,

$$= a \cup (b \cup c \cap c) \cap c.$$

Finally, by virtue of the definition of the analytic conjunction, this

$$= a \cup b \cap c,$$

the first expression being equal to the last. Expressing this result in words, and combining it with the previous results, we have then:

If the synthetic conjunction is elementary, then the order in which one synthetically or analytically conjoins is unimportant for the product. One can also insert or omit parentheses after a synthetic symbol if the conjunction contains only synthetic factors; but after an analytic symbol parentheses can be inserted or omitted only if the symbols preceding the factors in the *parentheses are simultaneously interchanged, that is, the analytic symbols are changed into synthetic and vice versa.*

This is the most general result we can obtain from the assumptions given. We have yet to establish that one can omit parentheses that follow a synthetic symbol and enclose an analytic symbol. This requires a new assumption.

§ 6. Uniqueness of Analysis; Addition and Subtraction

The new assumption we add is that the product of the analytic conjunction is unique, or in other words that if one factor of the synthetic conjunction remains fixed while the other changes, then the product always changes as well. From this, results

$$a \cap b \cup b = a;$$

now $a \cap b \cup b$ denotes the form that gives $a \cap b$ when synthetically conjoined with b. But a is such a form, and by virtue of the uniqueness of the result, the only one; the validity of the above equation is thus established. It then follows as well that

$$a \cap (b \cup c) = a \cap b \cup c.$$

To show that the right side of this equation equals the left, we replace b with $((b \cup c) \cap c)$, thus:

$$a \cap b \cup c = a \cap ((b \cup c) \cap c) \cup c;$$

according to § 4 this is

$$= a \cap (b \cup c) \cap c \cup c,$$

and again according to the theorem just established, this

$$= a \cap (b \cup c),$$

thus establishing that in fact the right side of the equation equals the left. Now since one can repeat this procedure when several factors are found in parentheses, one has the theorem:

If the synthetic conjunction is elementary and the corresponding analytic conjunction is unique, then one can insert or omit parentheses after a synthetic symbol. In this case (provided that that uniqueness is generally valid), we call the synthetic conjunction addition *and the corresponding analytic conjunction* subtraction.

As for the order of factors, one finds $a \cap b \cup c = a \cup c \cap b$; for $a \cap b \cup c = b \cap a \cup c = b \cap (a \cup c) = a \cup c \cap b$, whence we have also proved the interchangeability of two factors, one of which is preceded by a synthetic, the other by an analytic symbol, provided the uniqueness of the analytic product is assumed. And the theorems of this paragraph are only true under this assumption, while those of the preceding paragraph remain valid if the product of the analytic conjunction is multivalued.*

§ 7. The Indifferent and Analytic Forms

One is led by the analytic process to the indifferent and analytic forms. One obtains the former by the analytic conjunction of two equal forms; thus $a \cup a$ represents the *indifferent* form, which is in fact independent of the value a. Thus one has $a \cup a = b \cup b$, since $b \cup b$ represents the form that gives b when synthetically conjoined with b; $a \cup a$ is such a form since $b \cap (a \cup a) = b \cap a \cup a =$ b. In those circumstances where the analytic conjunction is unique, $a \cup a$ must therefore be set equal to $b \cup b$. Since the indifferent form always has a unique value under the assumptions made, there follows the necessity to mark it with its own symbol. For the present let us take this symbol to be $\sim$, and designate the form $(\sim \cup a)$ as $(\cup a)$, calling $(\cup a)$ the pure analytic form; in fact, if the synthetic conjunction is addition, it is the *negative* form. It follows directly that $(a \cap \sim)$ and $(a \cup \sim)$ are both equal to a, and further that $\cap (\cup a)$ equals $\cup a$ and $\cup (\cup a)$ equals $\cap a$, since one need only substitute the complete expressions for the forms just given and the validity of these equations becomes obvious.** The form analytic with respect to addition we call

*Examples of such multivaluedness appear not only in great abundance in extension theory, as will be shown later, but also in arithmetic, so the distinction is important there as well. In particular, addition and multiplication are taken as elementary conjunctions; and while subtraction is always unique, division is only unique so long as zero does not appear as divisor. For that reason only the theorems of the previous paragraph apply to division in general, whereas the theorems of this paragraph are only valid under the restriction that zero never appear as divisor. Failure to observe this condition necessarily results in the most awful contradictions and embarrassments, as has indeed already happened on occasion.

(1877) A subsequent investigation, establishing the law for the conjunction of multivalued magnitudes, has convinced me that in general one must always transform the multivalued magnitudes into single-valued ones before one can employ any of the theorems on conjunction. I have incorporated this conjecture into my *Ausdehnungslehre* of 1862 in the notes to nos. 348 and 477, and at the same time in the first of these showed how one can transform multivalued magnitudes into single-valued ones. This conjecture is also among the fundamentals in my *Arithmetik* (Stettin: 1860, printed and published by R. Grassmann [since 1861 by Enslin in Berlin as well]).

**It is a futile undertaking if, for example, after having established the laws for addition and subtraction for positive numbers in arithmetic, one then attempts to set up another set of laws especially for negative numbers. In particular, if one defines the negative number which, upon addition to a, gives zero, then by "addition" (this concept having originally been defined only for positive numbers) one now means either the same method of conjunction for which the fundamental laws (which are defined by the general concept of addition) are true, or something else. In the former case the demonstration is unnecessary, since the additional laws have then already been proved for the negative numbers; the latter case is impossible unless the concept of the addition of such numbers is not perhaps to be defined yet otherwise. Precisely the same circumstance obtains with fractions as with integers.

the negative form, and the indifferent form with respect to addition we call *zero.*

§ 8. Addition and Subtraction of Similar Forms

So far we have developed the concept of addition in a purely formal manner, since we have defined it from the validity of certain laws of conjunction. This formal concept also remains the only general one. Yet it is not the way we arrive at this concept in the individual branches of mathematics. Rather, in them a characteristic method of conjunction is obtained from the generation of the magnitudes itself, which is then interpreted as addition in precisely the general sense given, since those formal laws apply to it.

Specifically, if we consider two magnitudes (forms) that result from the continuation of the same method of generation, and which we therefore call "generated in the same sense," then it is clear that one may arrange them so that the two of them comprise a single whole, in such a way that their total content, that is the parts comprising the two of them, become one, correlated in thought. This whole may then be regarded as generated in the same sense as those two magnitudes. Now it is easy to show that this conjunction is an addition, that is that it is elementary and that its analysis is unique. First, I can arbitrarily combine and interchange, since the parts which are correlated in thought remain the same thereby, and their result cannot change since they are all alike (being produced by like generations). Its analysis is also unique; for were this not the case, then while the product and one factor of the synthetic conjunction remained fixed, the other factor could assume a different value. But one of these two values would then necessarily be greater than the other, whence yet more parts must have been added to it. But then the same parts must have been added to the product, whence the product would then have changed, contrary to assumption. Thus since the corresponding analytic form is unique, the synthetic conjunction is interpreted as an addition, and for these conjunctions all the laws of §§ 3–7 are valid. It therefore follows that the laws of this conjunction also remain unchanged when the factors are negative. If we compare the negative magnitudes with the positive, we can say that they are generated *in the opposite sense;* and we can combine both like and oppositely generated magnitudes under the name *similar* magnitudes, whence in this way the real concepts of addition and subtraction are defined for similar magnitudes in general.

§ 9. Conjunctions of Different Order. Multiplication

So far we have considered only one method of synthetic conjunction and its relation to the corresponding analytic conjunction. Now we turn to establishing the relation between two different methods of synthetic conjunction. To this end the one must be specified through the other in accordance with its definition.[6] This definition depends on the process by which an expression containing both types of conjunction can be transformed without altering the combined product.

The simplest way that both conjunctions can appear in a single expression is that in which the product of one conjunction is subjected to the other. Thus if $\cap$ and $\cap'$ are the symbols for the two conjunctions, then the relation between them depends on the transformations assumed to be permitted in the expression $(a \cap b)\cap' c$. If the second conjunction should be symmetrically related to the two factors of the first, then the simplest transformation is that in which one subjects each factor of the first conjunction to the second, and then applies these individual products as factors in the first method of conjunction. If this transformation can be undertaken without altering the combined product, that is if

$$(a \cap b) \cap' c = (a \cap' c) \cap (b \cap' c),$$

then we call the second a *conjunction of the next higher order* than the first.

In particular, if both factors of this second conjunction depend on the first in the same way, so that that definition of the new combination is valid for both the prefactor and the postfactor, and if in addition the first conjunction is elementary and its corresponding analysis unique, then we call the second conjunction *multiplication* since we have already adopted the name *addition* for the first. This is generally the way that initially, that is when no species of conjunction is yet given, such a conjunction of next higher order is defined. We therefore regard addition as the conjunction of first order, multiplication as that of second order.*

From now on we agree to replace the general conjunctive symbols with the usual conventions for these species of conjunctions, and indeed for multiplication we choose simple juxtaposition.

§ 10. General Laws of Multiplication

We have specified the relation of multiplication to addition as

$$(a + b)c = ac + bc$$
$$c(a + b) = ca + cb;$$

*On the same principle one can take that of third order to be raising to a power, which however we omit here for brevity. But it is in the nature of the subject that the definition for these conjunctions is only formal, and cannot be embodied by real definitions except in the individual sciences.

and we have thereby fixed the definition of multiplication. By repeated application of these fundamental laws, one is led at once to the more general theorem that if both factors are expanded out, then each term of the one can be multiplied by each term of the other, and these products can then be added. From this follows the corresponding law for the relation of multiplication to subtraction, that is

$$(a - b)c = ac - bc.$$

That is, to reduce the second member to the first, replace the a in it with its equal $(a - b) + b$ to get

$$ac - bc = ((a - b) + b)c - bc\,;$$

but, according to the laws established previously, the second member

$$= (a - b)c + bc - bc.$$

Now according to § 6 this expression

$$= (a - b)c,$$

whence the first expression equals the last. If the second factor is a difference the corresponding law follows in the same way. By repeated application of these laws one is led to the more general theorem:

If the factors of a product are constructed by additions and subtractions, then without altering the combined product one can multiply each term of the one by each term of the other and then conjoin the resulting products by prefixing addition and subtraction symbols, according as the symbols preceding the terms multiplied are the same or not.

§ 11. Laws of Division

For division, the law for the expansion of the dividend is generally valid, whether the result is unique or not;* that is

$$\frac{a \pm b}{c} = \frac{a}{c} \pm \frac{b}{c},$$

where, since the interchangeability of factors for multiplication cannot be assumed in general, two types of division must be distinguished, according as the prefactor or the postfactor of the multiplicative conjunction is sought. Since in either case both factors have the same relation to addition and subtraction, the above is valid for both types of division; and if the

*(1877) Cf. footnotes * and ** on p. 25 and nos. 377 to 391 of the *Ausdehnungslehre* of 1862.

above law is established for one type, then on the same grounds it is also established for the other.

We will assume that the prefactor is sought; thus for example, if

$$\frac{a}{\cdot c} = x,\text{* then } xc = a.$$

By $\frac{a+b}{\cdot c}$ is meant that form which, when multiplied by c, as prefactor, gives $a + b$. First, I can decompose each form into two parts, one of which may be assumed arbitrary. Thus let us replace $\frac{a+b}{\cdot c}$ with the equivalent form $\frac{a}{\cdot c} + x$. Multiply this by c, as prefactor, which according to the preceding paragraph yields $a + xc$; since however with this multiplication we must get $a + b$, we find

$$a + xc = a + b,$$

that is

$$xc = b, \; x = \frac{b}{\cdot c}.$$

Hence, since it was set equal to $\frac{a}{\cdot c} + x$, the form sought equals

$$\frac{a}{\cdot c} + \frac{b}{\cdot c}.$$

The law for differences is derived in the same way.

§ 12. Real Concept of Multiplication

The laws presented in the preceding paragraphs express the general relation of multiplication and division to addition and subtraction. However the laws of multiplication as such, as arithmetic formulates them, which express the interchangeability and combinability of its factors, do not result from this general concept of multiplication. On the contrary, in our science we may learn of other types of multiplication in which at least the interchangeability of the factors does not obtain, but in which all the other theorems presented have their full application.

We have therefore formally defined the general concept of this multiplication as well; if the nature of the magnitudes so conjoined is given, then this formal concept must correspond to a real concept that expresses the method of generation of the product by the factors. The relation to real addition provides us with a general definition of this method of genera-

*Here the dot in the divisor indicates the position of the factor sought.

tion. Thus if one of the factors is taken as the sum of its terms (according to § 8), then according to the general law relating multiplication to addition, instead of the sum being subjected to the product-forming method of generation, its terms can themselves be so subjected, and the products so formed can then be added; that is, since these products again appear as generated in the same sense, they can be conjoined as parts of a single whole. That is, the multiplicative method of generation must be such that all the terms of the factors enter it in the same way, so that if a term of the one, conjoined multiplicatively with a term of the other, generates a given magnitude, then in the multiplicative conjunction of the wholes each term of the former taken with each term of the latter generates such a magnitude, and indeed the *same* magnitude, that these terms taken together were originally assumed to equal. It is therefore immediately clear that if the method of generation has the given characteristics, then the method of conjunction corresponding to it has the multiplicative relation to the addition of similars, and consequently all the laws of that relation hold for it.

We therefore also call such a method of conjunction a *multiplication,* provided only that its multiplicative relation to addition is demonstrated, or in other words, provided only that the equal entry of all the terms of the conjunctive factors into the conjunction is established in the above sense.

The development of the general conjunctive laws to this point essentially suffices for the presentation of our science, with which we therefore now proceed.